

Generalized Derivations of Hom-Lie Superalgebras *

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Abstract

In this paper, we give some basic properties of the generalized derivation algebra $\text{GDer}(L)$ of a Hom-Lie superalgebra L . In particular, we prove that $\text{GDer}(L) = \text{QDer}(L) + \text{QC}(L)$, the sum of the quasiderivation algebra and the quasicentroid. We also prove that $\text{QDer}(L)$ can be embedded as derivations in a larger Hom-Lie superalgebra.

Key words: Hom-Lie superalgebras; Generalized derivations; Quasiderivations; Centroids; Quasicentroids.

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§0 Introduction

Hom-Lie algebras are a generalization of Lie algebras, where the classical Jacobi identity is twisted by a linear map. In the particular case that the twisted map is the identity map, Hom-Lie algebras become Lie algebras. The notion of Hom-Lie algebras was introduced by Hartwig, Larsson and Silvestrov to describe the structures on certain deformations of the Witt algebras and the Virasoro algebras [8]. Hom-Lie algebras are also related to deformed vector fields, the various versions of the Yang-Baxter equations, braid group representations, and quantum groups [8, 16, 17]. Recently, Hom-Lie superalgebras were studied in [2, 4, 13]. More applications of the Hom-Lie algebras, Hom-algebras and Hom-Lie superalgebras can be found in [7, 11, 15].

As is well known, derivation and generalized derivation algebra are very important subjects both in the research of Lie algebras and Lie superalgebras. In the

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study of Levi factors in derivation algebra of nilpotent Lie algebras, the generalized derivations, quasiderivations, centroids, and quasiceintroids play key roles [5]. Melville dealt particularly with the centroids of nilpotent Lie algebras [14]. The most important and systematic research on the generalized derivation algebra of a Lie algebra was due to Leger and Luks [12]. In [5], some nice properties of the generalized derivation algebra and their subalgebras, for example, of the quasiderivation algebra and of the centroid have been obtained. In particular, they investigated the structure of the generalized derivation algebra and characterized the Lie algebras satisfying certain conditions. Meanwhile, they also pointed that there exist some connections between quasiderivations and cohomology of Lie algebras.

For the generalized derivation algebra of general non-associative algebras, the readers will be referred to [9, 10, 14, 18].

The purpose of this paper is to generalize some beautiful results in [5, 18] to the generalized derivation algebra of a Hom-Lie superalgebra. In this paper, we mainly study the derivation algebra $\text{Der}(L)$, the center derivation algebra $\text{ZDer}(L)$, the quasiderivation algebra $\text{QDer}(L)$, and the generalized derivation algebra $\text{GDer}(L)$ of a Hom-Lie superalgebra L .

We proceed as follows. Firstly we recall some basic definitions and propositions which will be used in what follows. Then we give some basic properties of the generalized derivation algebra and their Hom-subalgebras, show that the quasiceintroid of a Hom-Lie superalgebra is also a Hom-Lie superalgebra if only and if it is a Hom-associative superalgebra. Finally we prove that the quasiderivations of L can be embedded as derivations in a larger Hom-Lie superalgebra and obtain a direct sum decomposition of $\text{Der}(L)$ when the annihilator of L is equal to zero.

§1 Preliminaries

Throughout this paper \mathbf{K} is a field of characteristic zero and $\mathbb{Z}_2 = \{\bar{0}, \bar{1}\}$. A vector space V is said to be \mathbb{Z}_2 -graded if $V = V_{\bar{0}} \oplus V_{\bar{1}}$. An element $x \in V_{\gamma}$ ($\gamma \in \mathbb{Z}_2$) is said to be homogeneous of degree γ . For simplicity the degree of an element x is denoted by $|x|$. Let V and W be two \mathbb{Z}_2 -graded vectors spaces. A linear map $f : V \rightarrow W$ is said to be homogeneous of degree $\xi \in \mathbb{Z}_2$, if $f(x)$ is homogeneous of degree $\gamma + \xi$ for all the element $x \in V_{\gamma}$. The set of all such maps is denoted by $\text{Hom}(V, W)_{\xi}$. It is a subspace of $\text{Hom}(V, W)$, the vector space of all linear maps from V into W . If in addition, f is homogeneous of degree $\bar{0}$, i.e. $f(V_{\gamma}) \subseteq V'_{\gamma}$ holds for any $\gamma \in \mathbb{Z}_2$, then f is said to be even.

Definition 1.1 [3] *A Hom-Lie superalgebra is a triple $(L, [\cdot, \cdot], \alpha)$ consisting of a \mathbb{Z}_2 -graded vector space $L = L_{\bar{0}} \oplus L_{\bar{1}}$, an even bilinear map $[\cdot, \cdot] : L \times L \rightarrow L$ (i.e. $[L_{\theta}, L_{\mu}] \subseteq L_{\theta+\mu}$) and an even linear map $\alpha : L \rightarrow L$ such that for homogeneous elements $x, y, z \in L$ we have*

- (1) $[x, y] = -(-1)^{xy}[y, x]$,
- (2) $(-1)^{zx}[\alpha(x), [y, z]] + (-1)^{xy}[\alpha(y), [z, x]] + (-1)^{yz}[\alpha(z), [x, y]] = 0$.

In particular, if α preserves the bracket, (i.e. $\alpha[x, y] = [\alpha x, \alpha y]$, $\forall x, y \in L$), then we call $(L, [\cdot, \cdot], \alpha)$ a multiplicative Hom-Lie superalgebra.

Definition 1.2 [3] *Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie superalgebra and define the fol-*

lowing subvector space \mathfrak{U} of $\text{End}(L)$ consisting of even linear maps u on L as follows:

$$\mathfrak{U} = \{u \in \text{End}(L) \mid u\alpha = \alpha u\}$$

and

$$\sigma : \mathfrak{U} \rightarrow \mathfrak{U}; \sigma(u) = \alpha u.$$

Then \mathfrak{U} is a Hom-Lie superalgebra over \mathbf{K} with the bracket

$$[D_\theta, D_\mu] = D_\theta D_\mu - (-1)^{\theta\mu} D_\mu D_\theta$$

for all $D_\theta, D_\mu \in \text{hg}(\mathfrak{U})$.

Definition 1.3 [3] Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. A homogeneous bilinear map $D : L \rightarrow L$ of degree d is said to be an α^k -derivation of L , where $k \in \mathbf{N}$, if it satisfies

$$D\alpha = \alpha D,$$

$$[D(x), \alpha^k(y)] + (-1)^{dx} [\alpha^k(x), D(y)] = D([x, y]),$$

$\forall x \in \text{hg}(L), y \in L$.

We denote the set of all α^k -derivations by $\text{Der}_{\alpha^k}(L)$, then $\text{Der}(L) := \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(L)$ provided with the super-commutator and the following even map

$$\tilde{\alpha} : \text{Der}(L) \rightarrow \text{Der}(L); \tilde{\alpha}(D) = D\alpha$$

is a Hom-subalgebra of \mathfrak{U} and is called the derivation algebra of L .

Definition 1.4 [1] An endomorphism $D \in \text{Der}_\theta(L)$ is said to be a homogeneous generalized α^k -derivation of degree θ of L , if there exist two endomorphisms $D', D'' \in \text{End}_\theta(L)$ such that

$$\begin{aligned} D\alpha = \alpha D = 0, \quad D\alpha' = \alpha' D = 0, \quad D\alpha'' = \alpha'' D = 0 \\ [D(x), \alpha^k(y)] + (-1)^{\theta x} [\alpha^k(x), D'(y)] = D''([x, y]), \end{aligned} \quad (1.1)$$

for all $x \in \text{hg}(L), y \in L$.

Definition 1.5 [1] An endomorphism $D \in \text{End}_\theta(L)$ is said to be a homogeneous α^k -quasiderivation of degree θ , if there exists an endomorphism $D' \in \text{End}_\theta(L)$ such that

$$\begin{aligned} D\alpha = \alpha D = 0, \quad D\alpha' = \alpha' D = 0, \\ [D(x), \alpha^k(y)] + (-1)^{\theta x} [\alpha^k(x), D'(y)] = D'([x, y]), \end{aligned} \quad (1.2)$$

for all $x \in \text{hg}(L), y \in L$.

Let $\text{GDer}_{\alpha^k}(L)$ and $\text{QDer}_{\alpha^k}(L)$ be the sets of homogeneous generalized α^k -derivations and of homogeneous α^k -quasiderivations, respectively. That is

$$\text{GDer}(L) := \bigoplus_{k \geq 0} \text{GDer}_{\alpha^k}(L), \quad \text{QDer}(L) := \bigoplus_{k \geq 0} \text{QDer}_{\alpha^k}(L).$$

It is easy to verify that both $\text{GDer}(L)$ and $\text{QDer}(L)$ are Hom-subalgebras of \mathfrak{U} (see Proposition 2.1).

Definition 1.6 [1] If $C(L) := \bigoplus_{k \geq 0} C_{\alpha^k}(L)$, with $C_{\alpha^k}(L)$ consisting of $D \in \text{hg}(\text{End}(L))$ of degree d satisfying

$$D\alpha = \alpha D,$$

$$[D(x), \alpha^k(y)] = (-1)^{dx} [\alpha^k(x), D(y)] = D([x, y]),$$

for all $x \in \text{hg}(L), y \in L$, then $C(L)$ is called an α^k -centroid of L .

Definition 1.7 [1] If $\text{QC}(L) := \bigoplus_{k \geq 0} \text{QC}_{\alpha^k}(L)$ with $\text{QC}_{\alpha^k}(L)$ consisting of $D \in \text{hg}(\text{End}(L))$ of degree d such that

$$[D(x), \alpha^k(y)] = (-1)^{dx} [\alpha^k(x), D(y)],$$

for all $x \in \text{hg}(L), y \in L$, then $\text{QC}(L)$ is called an α^k -quasicentroid of L .

Define $\text{ZDer}(L) := \bigoplus_{k \geq 0} \text{Der}_{\alpha^k}(L)$, where $\text{Der}_{\alpha^k}(L)$ consists of $D \in \text{hg}(\text{End}(L))$ such that

$$[D(x), \alpha^k(y)] = D([x, y]) = 0,$$

for all $x \in \text{hg}(L), y \in L$.

It is easy to verify that

$$\text{ZDer}(L) \subseteq \text{Der}(L) \subseteq \text{QDer}(L) \subseteq \text{GDer}(L) \subseteq \text{Pl}(L).$$

$$C(L) \subseteq \text{QC}(L) \subseteq \text{QDer}(L).$$

Definition 1.8 [3] Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. If $Z(L) := \bigoplus_{\theta \in \Gamma} Z_{\theta}(L)$, with $Z_{\theta}(L) = \{x \in L_{\theta} \mid [x, y] = 0, \forall x \in \text{hg}(L), y \in L\}$, then $Z(L)$ is called the center of L .

§2 Generalized derivation algebra and their Hom-subalgebras

First, we give some basic properties of center derivation algebra, quasiderivation algebra and the generalized derivation algebra of a Hom-Lie superalgebra.

Proposition 2.1 Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. Then the following statements hold:

- (1) $\text{GDer}(L), \text{QGer}(L)$ and $C(L)$ are Hom-subalgebras of \mathfrak{U} .
- (2) $\text{ZDer}(L)$ is a Hom-ideal of $\text{Der}(L)$.

Proof. (1) Assume that $D_{\theta} \in \text{GDer}_{\alpha^k}(L)$, $D_{\mu} \in \text{GDer}_{\alpha^s}(L)$, $\forall x \in \text{hg}(L)$ and $y \in L$. We have

$$\begin{aligned} [(\tilde{\alpha}(D_{\theta}))(x), \alpha^{k+1}(y)] &= [(D_{\theta}\alpha)(x), \alpha^{k+1}(y)] = \alpha[D_{\theta}(x), \alpha^k(y)] \\ &= \alpha(D_{\theta}''([x, y]) - (-1)^{\theta x} [\alpha^k(x), D'_{\theta}(y)]) \\ &= \tilde{\alpha}(D_{\theta}'')([x, y]) - (-1)^{\theta x} [\alpha^{k+1}(x), \tilde{\alpha}(D'_{\theta})(y)]. \end{aligned}$$

Since both $\tilde{\alpha}(D_{\theta}'')$ and $\tilde{\alpha}(D'_{\theta})$ are in $\text{End}_{\theta}(L)$, $\tilde{\alpha}(D_{\theta}) \in \text{GDer}_{\alpha^{k+1}}(L)$ of degree θ .

We also have

$$\begin{aligned} [D_{\theta}D_{\mu}(x), \alpha^{k+s}(y)] &= D_{\theta}''D_{\mu}''([x, y]) + (-1)^{\theta(\mu+x)}(-1)^{\mu x} [\alpha^{s+k}(x), D'_{\mu}D'_{\theta}(y)] \\ &\quad - (-1)^{\theta(\mu+x)} D_{\mu}''([\alpha^k(x), D'_{\theta}(y)]) - (-1)^{\mu x} D_{\theta}''([\alpha^s(x), D'_{\mu}(y)]) \end{aligned}$$

and

$$\begin{aligned} [D_\mu D_\theta(x), \alpha^{k+s}(y)] &= D_\mu'' D_\theta''([x, y]) + (-1)^{\mu(\theta+x)} (-1)^{(\theta x)} [\alpha^{s+k}(x), D_\theta' D_\mu'(y)] \\ &\quad - (-1)^{\mu(\theta+x)} D_\theta''([\alpha^s(x), D_\mu'(y)]) - (-1)^{\theta x} D_\mu''([\alpha^k(x), D_\theta'(y)]). \end{aligned}$$

Thus for all $x \in \text{hg}(L)$ and $y \in L$, we have

$$[[D_\theta, D_\mu](x), \alpha^{k+s}(y)] = [D_\theta'', D_\mu'']([x, y]) - (-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), [D_\theta', D_\mu'](y)].$$

Since both $[D_\theta', D_\mu']$ and $[D_\theta'', D_\mu'']$ are in $\text{End}_{\theta+\mu}(L)$, $[D_\theta, D_\mu] \in \text{GDer}_{\alpha^{k+s}}(L)$ of degree $\theta + \mu$, $\forall \theta, \mu \in G$, $\text{GDer}(L)$ is a Hom-subalgebra of \mathcal{U} .

Similarly, $\text{QGer}(L)$ is a Hom-subalgebra of \mathcal{U} .

Assume that $D_\theta \in C_{\alpha^k}(L)$, $D_\mu \in C_{\alpha^s}(L)$, $\forall x \in \text{hg}(L)$, $y \in L$. Then

$$\begin{aligned} [\tilde{\alpha}(D_\theta)(x), \alpha^{k+1}(y)] &= \alpha([D_\theta(x), \alpha^k(y)]) = (-1)^{\theta x} \alpha([\alpha^k(x), D_\theta(y)]) \\ &= (-1)^{\theta x} [\alpha^{k+1}(x), \tilde{\alpha}(D_\theta)(y)], \end{aligned}$$

and so

$$\tilde{\alpha}(D_\theta) \in C_{\alpha^{k+1}}(L).$$

Note that

$$\begin{aligned} [[D_\theta, D_\mu](x), \alpha^{k+s}(y)] &= [D_\theta D_\mu(x), \alpha^{k+s}(y)] - (-1)^{\theta \mu} [D_\mu D_\theta(x), \alpha^{k+s}(y)] \\ &= D_\theta([D_\mu(x), \alpha^s(y)]) - (-1)^{\theta \mu} D_\mu([D_\theta(x), \alpha^k(y)]) \\ &= D_\theta D_\mu([x, y]) - (-1)^{\theta \mu} D_\mu D_\theta([x, y]) \\ &= [D_\theta, D_\mu]([x, y]). \end{aligned}$$

Similarly,

$$(-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), [D_\theta, D_\mu](y)] = [D_\theta, D_\mu]([x, y]).$$

Then $[D_\theta, D_\mu] \in C_{\alpha^{k+s}}(L)$ of degree $\theta + \mu$, $\forall \theta, \mu \in G$. Thus $C(L)$ is a Hom-subalgebra of \mathcal{U} .

(2) Assume that $D_\theta \in \text{ZDer}_{\alpha^k}(L)$, $D_\mu \in \text{Der}_{\alpha^s}(L)$, $\forall x \in \text{hg}(L)$, $y \in L$. Then

$$[\tilde{\alpha}(D_\theta)(x), \alpha^{k+1}(y)] = \alpha([D_\theta(x), \alpha^k(y)]) = \alpha D_\theta([x, y]) = \tilde{\alpha}(D_\theta)([x, y]) = 0.$$

So

$$\tilde{\alpha}(D_\theta) \in \text{ZDer}_{\alpha^{k+1}}(L).$$

Note that

$$\begin{aligned} [[D_\theta, D_\mu]([x, y])] &= D_\theta D_\mu([x, y]) - (-1)^{\theta \mu} D_\mu D_\theta([x, y]) \\ &= D_\theta([D_\mu(x), \alpha^s(y)] + (-1)^{\mu x} [\alpha^s(x), D_\mu(x)]) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} [[D_\theta, D_\mu](x), \alpha^{s+k}(y)] &= [(D_\theta D_\mu - (-1)^{\theta \mu} D_\mu D_\theta)(x), \alpha^{s+k}(y)] \\ &= -(-1)^{\theta \mu} (D_\mu([D_\theta(x), \alpha^k(y)] - (-1)^{\mu(\theta+x)} [\alpha^s(D_\theta(x)), D_\mu(\alpha^k(y))]) \\ &= -(-1)^{\mu(\theta+x)} [D_\theta(\alpha^s(x)), \alpha^k(D_\mu(y))] = 0. \end{aligned}$$

Then $[D_\theta, D_\mu] \in \text{ZDer}_{\alpha^{k+s}}(L)$ of degree $\theta + \mu$, $\forall \theta, \mu \in G$. Thus $\text{ZDer}(L)$ is a Hom-ideal of $\text{Der}(L)$. \square

Lemma 2.2 *Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. Then*

- (1) $[\text{Der}(L), \text{C}(L)] \subseteq \text{C}(L)$.
- (2) $[\text{QDer}(L), \text{QC}(L)] \subseteq \text{QC}(L)$.
- (3) $[\text{QC}(L), \text{QC}(L)] \subseteq \text{QDer}(L)$.
- (4) $\text{C}(L) \subseteq \text{QDer}(L)$.

Proof. (1) Assume that $D_\theta \in \text{GDer}_{\alpha^k}(L)$, $D_\mu \in \text{C}_{\alpha^s}(L)$, $\forall x \in \text{hg}(L)$ and $y \in L$. We have

$$\begin{aligned} [D_\theta D_\mu(x), \alpha^{k+s}(y)] &= D_\theta([D_\mu(x), \alpha^s(y)]) - (-1)^{\theta(\mu+x)}[\alpha^k(D_\mu(x)), D_\theta(\alpha^s(y))] \\ &= D_\theta([D_\mu(x), \alpha^s(y)]) - (-1)^{\theta(\mu+x)}[D_\mu(\alpha^k(x)), \alpha^s(D_\theta(y))] \\ &= D_\theta D_\mu([x, y]) - (-1)^{\theta(\mu+x)}(-1)^{\mu x}[\alpha^{k+s}(x), D_\mu D_\theta(y)], \end{aligned}$$

and

$$\begin{aligned} [D_\mu D_\theta(x), \alpha^{k+s}(y)] &= D_\mu([D_\theta(x), \alpha^k(y)]) \\ &= D_\mu D_\theta([x, y]) - (-1)^{\theta x} D_\mu([\alpha^k(x), D_\theta(y)]) \\ &= D_\mu D_\theta([x, y]) - (-1)^{(\theta+\mu)x}[\alpha^{k+s}(x), D_\mu D_\theta(y)]. \end{aligned}$$

Hence,

$$\begin{aligned} [[D_\theta, D_\mu](x), \alpha^{k+s}(y)] &= [D_\theta D_\mu(x), \alpha^{k+s}(y)] - (-1)^{\theta\mu} [D_\mu D_\theta(x), \alpha^{k+s}(y)] \\ &= D_\theta D_\mu([x, y]) - (-1)^{\theta\mu} D_\mu D_\theta([x, y]) \\ &= [D_\theta, D_\mu]([x, y]). \end{aligned}$$

On the other hand,

$$\begin{aligned} [D_\theta D_\mu(x), \alpha^{k+s}(y)] &= D_\theta([D_\mu(x), \alpha^s(y)]) - (-1)^{\theta(\mu+x)}[\alpha^k(D_\mu(x)), D_\theta(\alpha^s(y))] \\ &= (-1)^{\mu x} (D_\theta([\alpha^s(x), D_\mu(y)]) - (-1)^{\theta(\mu+x)}[\alpha^{k+s}(x), D_\mu D_\theta(y)]) \\ &= (-1)^{\mu x} [D_\theta(\alpha^s(x)), \alpha^k(D_\mu(y))] + (-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), D_\theta D_\mu(y)] \\ &\quad - (-1)^{\theta\mu} (-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), D_\mu D_\theta(y)], \\ [D_\mu D_\theta(x), \alpha^{k+s}(y)] &= (-1)^{\mu(\theta+x)} [D_\theta(\alpha^s(x)), \alpha^k(D_\mu(y))]. \end{aligned}$$

Then

$$\begin{aligned} [[D_\theta, D_\mu](x), \alpha^{k+s}(y)] &= [D_\theta D_\mu(x), \alpha^{k+s}(y)] - (-1)^{\theta\mu} [D_\mu D_\theta(x), \alpha^{k+s}(y)] \\ &= (-1)^{(\theta+\mu)x} ([\alpha^{k+s}(x), D_\theta D_\mu(y)] - (-1)^{\theta\mu} [\alpha^{k+s}(x), D_\mu D_\theta(y)]) \\ &= (-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), [D_\theta, D_\mu](y)]. \end{aligned}$$

Thus $[D_\theta, D_\mu] \in \text{C}_{\alpha^{s+k}}(L)$ of degree $\theta + \mu$, and we get $[\text{Der}(L), \text{C}(L)] \subseteq \text{C}(L)$.

(2) Similar to the proof of (1).

(3) Assume that $D_\theta \in \text{QC}_{\alpha^k}(L)$, $D_\mu \in \text{C}_{\alpha^s}(L)$, $\forall x \in \text{hg}(L)$ and $y \in L$. By Proposition 5.3 in [1], we have

$$[[D_\theta, D_\mu](x), \alpha^{k+s}(y)] + (-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), [D_\theta, D_\mu](y)] = 0.$$

Let $D' = 0$, hence $[D_\theta, D_\mu] \in \text{QDer}_{\alpha^{k+s}}(L)$ of degree $\theta + \mu$ as desired.

(4) See Proposition 5.2 in [1]. □

Theorem 2.3 *Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. Then*

$$\text{GDer}(L) = \text{QDer}(L) + \text{QC}(L).$$

Proof. Let $D_\theta \in \text{GDer}_{\alpha^k}(L)$. Then for all $x, y \in \text{hg}(L)$, there exist $D'_\theta, D''_\theta \in \text{End}_\theta(L)$ such that

$$[D_\theta(x), \alpha^k(y)] + (-1)^{\theta x}[\alpha^k(x), D'_\theta(y)] = D''_\theta([x, y]).$$

Since $(-1)^{(\theta+x)y}[\alpha^k(y), D_\theta(x)] + (-1)^{xy}[D'_\theta(y), \alpha^k(x)] = (-1)^{xy}D''_\theta([y, x])$,

$$[D'_\theta(y), \alpha^k(x)] + (-1)^{\theta y}[\alpha^k(y), D_\theta(x)] = D''_\theta([y, x]).$$

Hence $D'_\theta \in \text{GDer}_{\alpha^k}(L)$. For all $x, y \in \text{hg}(L)$, we have

$$[\frac{D_\theta + D'_\theta}{2}(x), \alpha^k(y)] + (-1)^{\theta x}[\alpha^k(x), \frac{D_\theta + D'_\theta}{2}(y)] = D''_\theta([x, y]),$$

and

$$[\frac{D_\theta - D'_\theta}{2}(x), \alpha^k(y)] - (-1)^{\theta x}[\alpha^k(x), \frac{D_\theta - D'_\theta}{2}(y)] = 0,$$

which imply that $\frac{D_\theta + D'_\theta}{2} \in \text{QDer}_{\alpha^k}(L)$ and $\frac{D_\theta - D'_\theta}{2} \in \text{QC}_{\alpha^k}(L)$, of degree θ . Hence

$$D_\theta = \frac{D_\theta + D'_\theta}{2} + \frac{D_\theta - D'_\theta}{2} \in \text{QDer}(L) + \text{QC}(L),$$

and

$$\text{GDer}(L) \subseteq \text{QDer}(L) + \text{QC}(L).$$

It is easy to verify that $\text{QDer}(L) + \text{QC}(L) \subseteq \text{GDer}(L)$. Therefore $\text{QDer}(L) + \text{QC}(L) = \text{GDer}(L)$. \square

Theorem 2.4 *Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra, α a surjection and $Z(L)$ the center of L . Then $[\text{C}(L), \text{QC}(L)] \subseteq \text{End}(L, Z(L))$. Moreover, if $Z(L) = \{0\}$, then $[\text{C}(L), \text{QC}(L)] = \{0\}$.*

Proof. Assume that $D_\theta \in \text{C}_{\alpha^k}(L)$, $D_\mu \in \text{QC}_{\alpha^s}(L)$ and $x \in \text{hg}(L)$. Since α is surjection, $\forall y' \in L$, $\exists y \in L$, such that $y' = \alpha^{k+s}(y)$. Then

$$\begin{aligned} [[D_\theta, D_\mu](x), y'] &= [[D_\theta, D_\mu](x), \alpha^{k+s}(y)] \\ &= [D_\theta D_\mu(x), \alpha^{k+s}(y)] - (-1)^{\theta \mu} [D_\mu D_\theta(x), \alpha^{k+s}(y)] \\ &= D_\theta([D_\mu(x), \alpha^s(y)]) - (-1)^{\mu x} [\alpha^s D_\theta(x), D_\mu \alpha^k(y)] \\ &= D_\theta([D_\mu(x), \alpha^s(y)]) - (-1)^{\mu x} D_\theta([\alpha^s(x), D_\mu(y)]) \\ &= D_\theta([D_\mu(x), \alpha^s(y)] - (-1)^{\mu x} [\alpha^s(x), D_\mu(y)]) = 0. \end{aligned}$$

Hence $[D_\theta, D_\mu](x) \in Z(L)$, and $[D_\theta, D_\mu] \in \text{End}(L, Z(L))$ as desired. Furthermore, if $Z(L) = \{0\}$, it is clear that $[\text{C}(L), \text{QC}(L)] = \{0\}$. \square

By Theorem 2.3 in [6], if $(L, [\cdot, \cdot])$ is a Lie superalgebra with $Z(L) = \{0\}$, where $Z(L)$ is the center of L , then $\text{C}(L) = \text{QDer}(L) \cap \text{QC}(L)$. But it is not true in case that $(L, [\cdot, \cdot], \alpha)$ is a multiplicative Hom-Lie superalgebra.

Example 2.5 Let $\{x_1, x_2, x_3\}$ be a basis of a 3-dimensional linear space L over \mathbf{K} . The following bracket and linear map α on L define a Hom-Lie algebra over \mathbf{K} :

$$\begin{aligned} [x_1, x_2] &= x_1, & \alpha(x_1) &= x_1, \\ [x_1, x_3] &= x_2, & \alpha(x_2) &= 2x_2, \\ [x_2, x_3] &= 2x_3, & \alpha(x_3) &= 2x_3, \end{aligned}$$

with $[x_2, x_1], [x_3, x_1], [x_3, x_2]$ defined via skewsymmetry.

Define $D : L \rightarrow L$ satisfying

$$D(x_1) = x_1, \quad D(x_2) = 2^k x_2, \quad D(x_3) = 2^k x_3. \quad (k \in \mathbf{Z}_+)$$

It is obvious that $Z(L) = 0$. $\forall y \in L$, suppose $y = ax_1 + bx_2 + cx_3$. Define $D' \in \text{End}(L)$ by

$$D'(x_1) = 2^{k+1} x_1, \quad D'(x_2) = 2^{k+1} x_2, \quad D'(x_3) = 2^{k+1} 2^k x_3.$$

It is obvious that for $i = 1, 2, 3$,

$$D'([x_i, ax_1 + bx_2 + cx_3]) = [D(x_i), \alpha^k(ax_1 + bx_2 + cx_3)] + [\alpha^k(x_i), D(ax_1 + bx_2 + cx_3)],$$

and

$$[D(x_i), \alpha^k(ax_1 + bx_2 + cx_3)] = [\alpha^k(x_i), D(ax_1 + bx_2 + cx_3)].$$

But for all $t \in \mathbf{Z}$,

$$D \in \text{QDer}(L) \cap \text{QC}(L).$$

While

$$D([x_1, ax_1 + bx_2 + cx_3]) = D(bx_1 + cx_2) = bx_1 + c2^k x_2.$$

$$[\alpha^t(x_1), D(ax_1 + bx_2 + cx_3)] = [x_1, ax_1 + b2^k x_2 + c2^k x_3] = b2^k x_1 + c2^k x_2.$$

That means $D \notin \text{C}(L)$.

Proposition 2.6 Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra and α a surjection. If $Z(L) = \{0\}$, then $\text{QC}(L)$ is a Hom-Lie superalgebra if and only if $[\text{QC}(L), \text{QC}(L)] = 0$.

Proof. (\Rightarrow) Assume that $D_\theta \in \text{hg}(\text{QC}_{\alpha^k}(L))$, $D_\mu \in \text{hg}(\text{QC}_{\alpha^s}(L))$, $x \in \text{hg}(L)$. Since α is a surjection, $\forall y' \in L$, $\exists y \in L$, such that $y' = \alpha^{k+s}(y)$. Since $\text{QC}(L)$ is a Hom-Lie superalgebra, $[D_\theta, D_\mu] \in \text{hg}(\text{QC}_{\alpha^{k+s}}(L))$, of degree $\theta + \mu$. Then

$$[[D_\theta, D_\mu](x), y'] = [[D_\theta, D_\mu](x), \alpha^{k+s}(y)] = (-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), [D_\theta, D_\mu](y)].$$

From the proof of Lemma 2.2 (1), we have

$$[[D_\theta, D_\mu](x), y'] = [[D_\theta, D_\mu](x), \alpha^{k+s}(y)] = -(-1)^{(\theta+\mu)x} [\alpha^{k+s}(x), [D_\theta, D_\mu](y)].$$

Hence $[[D_\theta, D_\mu](x), y'] = [[D_\theta, D_\mu](x), \alpha^{k+s}(y)] = 0$, i.e. $[D_\theta, D_\mu] = 0$.

(\Leftarrow) It is clear. □

Definition 2.7 [1] Let (L, μ, α) be a Hom-superalgebra.

(1) The Hom-associator of L is the trilinear map $as_\alpha : L \times L \times L \rightarrow L$ defined as

$$as_\alpha = \mu \circ (\mu \otimes \alpha - \alpha \otimes \mu).$$

In terms of elements, the map as_α is given by

$$as_\alpha(x, y, z) = \mu(\mu(x, y), \alpha(z)) - \mu(\alpha(x), \mu(y, z))$$

for all $x, y, z \in \text{hg}(L)$.

(2) Let L be a Hom-algebra over a field \mathbf{K} of characteristic $\neq 2$ with an even bilinear multiplication \circ . If L is Z_2 -graded and $\alpha : L \rightarrow L$ is an even linear map, then (L, \circ, α) is a Hom-Jordan superalgebra if the following identities

$$\begin{aligned} x \circ y &= (-1)^{xy} y \circ x, \\ (-1)^{z(x+w)} as_\alpha(x \circ y, \alpha(z), \alpha(w)) &+ (-1)^{x(y+z)} as_\alpha(y \circ w, \alpha(z), \alpha(x)) \\ &+ (-1)^{y(w+z)} as_\alpha(w \circ x, \alpha(z), \alpha(y)) = 0, \end{aligned}$$

hold for all $x, y, z \in \text{hg}(L)$.

Proposition 2.8 [1] (1) Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra, with the operation $D_\lambda \bullet D_\theta = D_\lambda D_\theta + (-1)^{\lambda\theta} D_\theta D_\lambda$, for all α -derivations $D_\lambda, D_\theta \in \text{hg}(\text{End}(L))$, the triple $(\text{End}(L), \bullet, \alpha)$ is a Hom-Jordan superalgebra.

(2) $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra, with the operation $D_\lambda \bullet D_\theta = D_\lambda D_\theta + (-1)^{\lambda\theta} D_\theta D_\lambda$, for all elements $D_\lambda, D_\theta \in \text{hg}(\text{QC}(L))$. Then the triple $(\text{QC}(L), \bullet, \alpha)$ is a Hom-Jordan superalgebra.

Theorem 2.9 Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra. Then $\text{QC}(L)$ is a Hom-Lie superalgebra with $[D_\lambda, D_\theta] = D_\lambda D_\theta - (-1)^{\lambda\theta} D_\theta D_\lambda$ if and only if $\text{QC}(L)$ is also a Hom-associative superalgebra with respect to composition.

Proof. (\Leftarrow) For all $D_\lambda, D_\theta \in \text{hg}(\text{QC}(L))$, we have $D_\lambda D_\theta \in \text{QC}(L)$ and $D_\theta D_\lambda \in \text{QC}(L)$, so $[D_\lambda, D_\theta] = D_\lambda D_\theta - (-1)^{\lambda\theta} D_\theta D_\lambda \in \text{QC}(L)$. Hence, $\text{QC}(L)$ is a Hom-Lie superalgebra.

(\Rightarrow) Note that $D_\lambda D_\theta = D_\lambda \bullet D_\theta + \frac{[D_\lambda, D_\theta]}{2}$ and by Proposition 2.8, we have $D_\lambda \bullet D_\theta \in \text{QC}(L)$, $[D_\lambda, D_\theta] \in \text{QC}(L)$. It follows that $D_\lambda D_\theta \in \text{QC}(L)$ as desired. \square

§3 The quasiderivations of Hom-Lie superalgebras

In this section, we will prove that the quasiderivations of L can be embedded as derivations in a larger Hom-Lie superalgebra and obtain a direct sum decomposition of $\text{Der}(L)$ when the annihilator of L is equal to zero.

Proposition 3.1 Let $(L, [\cdot, \cdot], \alpha)$ be a Hom-Lie superalgebra over \mathbf{K} and t an indeterminate. We define $L_g := L_g[tF[t]/(t^3)] = \{\Sigma(x_g \otimes t + y_g \otimes t^2) | x_g, y_g \in L_g\}$, $\check{\alpha}(\check{L}_g) := \{\Sigma(\alpha(x_g) \otimes t + \alpha(y_g) \otimes t^2) : x_g, y_g \in L_g\}$, and let $\check{L} = \check{L}_0 \oplus \check{L}_1$. Then \check{L} is a Hom-Lie superalgebra with the operation $[x_\lambda \otimes t^i, x_\theta \otimes t^j] = [x_\lambda, x_\theta] \otimes t^{i+j}$, for all $x_\lambda, x_\theta \in \text{hg}(L)$, $i, j \in \{1, 2\}$.

Proof. For all $x_\lambda, x_\theta, x_\mu \in \text{hg}(L)$ and $i, j, k \in \{1, 2\}$, we have

$$\begin{aligned} [x_\lambda \otimes t^i, x_\theta \otimes t^j] &= [x_\lambda, x_\theta] \otimes t^{i+j} \\ &= -(-1)^{\lambda\theta} [x_\theta, x_\lambda] \otimes t^{i+j} \\ &= -(-1)^{\lambda\theta} [x_\theta \otimes t^j, x_\lambda \otimes t^i], \end{aligned}$$

and

$$\begin{aligned}
& [\check{\alpha}(x_\lambda \otimes t^i), [x_\theta \otimes t^j, x_\mu \otimes t^k]] = [\alpha(x_\lambda), [x_\theta, x_\mu]] \otimes t^{i+j+k} \\
& = ([x_\lambda, x_\theta], \alpha(x_\mu)) + (-1)^{\lambda\theta} [\alpha(x_\theta), [x_\lambda, x_\mu]] \otimes t^{i+j+k} \\
& = [[x_\lambda, x_\theta], \alpha(x_\mu)] \otimes t^{i+j+k} + (-1)^{\lambda\theta} [\alpha(x_\theta), [x_\lambda, x_\mu]] \otimes t^{i+j+k} \\
& = [[x_\lambda \otimes t^i, x_\theta \otimes t^j], \check{\alpha}(x_\mu \otimes t^k)] + (-1)^{\lambda\theta} [\check{\alpha}(x_\theta \otimes t^j), [x_\lambda \otimes t^i, x_\mu \otimes t^k]].
\end{aligned}$$

Hence \check{L} is a Hom-Lie superalgebra. \square

For notational convenience, we write $xt(xt^2)$ in place of $x \otimes t(x \otimes t^2)$.

If U is a \mathbb{Z}_2 -graded subspace of L such that $L = U \oplus [L, L]$, then

$$\check{L} = Lt + Lt^2 = Lt + [L, L]t^2 + Ut^2,$$

or more precisely,

$$\check{L} = \check{L}_0 \oplus \check{L}_1 = (L_0t + [L, L]_0t^2 + U_0t^2) \oplus (L_1t + [L, L]_1t^2 + U_1t^2).$$

Now we define a map $\varphi : \text{QDer}(L) \rightarrow \text{End}(\check{L})$ satisfying

$$\varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2,$$

where D, D' satisfy (1.2), $a \in \text{hg}(L), b \in \text{hg}([L, L]), u \in \text{hg}(U)$ and $d(a) = d(b) = d(u)$.

Proposition 3.2 (1) $d(\varphi) = 0$.

(2) φ is injective and $\varphi(D)$ does not depend on the choice of D' .

(3) $\varphi(\text{QDer}(L)) \subseteq \text{Der}(\check{L})$.

Proof. (1) It is clear.

(2) If $\varphi(D_\lambda) = \varphi(D_\theta)$, then for all $a \in \text{hg}(L), b \in \text{hg}([L, L])$ and $u \in \text{hg}(U)$, we have

$$\varphi(D_\lambda)(at + bt^2 + ut^2) = \varphi(D_\theta)(at + bt^2 + ut^2),$$

that is

$$D_\lambda(a)t + D'_\lambda(b)t^2 = D_\theta(a)t + D'_\theta(b)t^2,$$

so $D_\lambda(a) = D_\theta(a)$. Hence $D_\lambda = D_\theta$, and φ is injective.

Suppose that there exists D'' such that

$$\varphi(D)(at + bt^2 + ut^2) = D(a)t + D''(b)t^2,$$

and

$$[D(x), \alpha^k(y)] + (-1)^{Dx} [\alpha^k(x), D(y)] = D''([x, y]),$$

then we have

$$D'([x, y]) = D''([x, y]),$$

thus $D'(b) = D''(b)$. Hence

$$\varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2 = D(a)t + D''(b)t^2,$$

which implies $\varphi(D)$ is determined by D .

(3) We have $[x_\lambda t^i, x_\theta t^j] = [x_\lambda, x_\theta] t^{i+j} = 0$, for all $i + j \geq 3$. Thus, to show $\varphi(D) \in \text{Der}(\check{L})$, we need only to check the validness of the following equation

$$\varphi(D)([xt, yt]) = [\varphi(D)(xt), \check{\alpha}^k(yt)] + (-1)^{Dx}[\check{\alpha}^k(xt), \varphi(D)(yt)].$$

For all $x, y \in \text{hg}(L)$, we have

$$\begin{aligned} \varphi(D)([xt, yt]) &= \varphi(D)([x, y]t^2) = D'([x, y])t^2 \\ &= ([D(x), \alpha^k(y)] + (-1)^{Dx}[\alpha^k(x), D(y)])t^2 \\ &= [D(x)t, \alpha^k(y)t] + (-1)^{Dx}[\alpha^k(x)t, D(y)t] \\ &= [\varphi(D)(xt), \check{\alpha}^k(yt)] + (-1)^{Dx}[\check{\alpha}^k(xt), \varphi(D)(yt)]. \end{aligned}$$

Therefore, for all $D \in \text{QDer}(L)$, we have $\varphi(D) \in \text{Der}(\check{L})$. \square

Proposition 3.3 *Let $(L, [\cdot, \cdot], \alpha)$ be a multiplicative Hom-Lie superalgebra and α a surjection. $Z(L) = \{0\}$ and \check{L} , φ are as defined above. Then $\text{Der}(\check{L}) = \varphi(\text{QDer}(L)) \oplus \text{ZDer}(\check{L})$.*

Proof. Since $Z(L) = \{0\}$, we have $Z(\check{L}) = Lt^2$. For all $g \in \text{Der}(\check{L})$, we have $g(Z(\check{L})) \subseteq Z(\check{L})$, hence $g(Ut^2) \subseteq g(Z(\check{L})) \subseteq Z(\check{L}) = Lt^2$. Now we define a map $f : Lt + [L, L]t^2 + Ut^2 \rightarrow Lt^2$ by

$$f(x) = \begin{cases} g(x) \cap Lt^2, & x \in Lt; \\ g(x), & x \in Ut^2; \\ 0, & x \in [L, L]t^2. \end{cases}$$

It is clear that f is linear. Note that

$$f([\check{L}, \check{L}]) = f([L, L]t^2) = 0, \quad [f(\check{L}), \check{\alpha}^k L] \subseteq [Lt^2, \alpha^k(L)t + \alpha^k(L)t^2] = 0,$$

hence $f \in \text{ZDer}(\check{L})$. Since

$$(g - f)(Lt) = g(Lt) - g(Lt) \cap Lt^2 = g(Lt) - Lt^2 \subseteq Lt, \quad (g - f)(Ut^2) = 0,$$

and

$$(g - f)([L, L]t^2) = g([L, L]t^2) \subseteq [\check{L}, \check{L}] = [L, L]t^2,$$

there exist $D, D' \in \text{End}(L)$ such that for all $a \in L, b \in [L, L]$,

$$(g - f)(at) = D(a)t, \quad (g - f)(bt^2) = D'(b)t^2.$$

Since $(g - f) \in \text{Der}(\check{L})$ and by the definition of $\text{Der}(\check{L})$, we have

$$[(g - f)(a_1t), \check{\alpha}^k(a_2t)] + (-1)^{(g-f)a_1}[\check{\alpha}^k(a_1t), (g - f)(a_2t)] = (g - f)([a_1t, a_2t]),$$

for all $a_1, a_2 \in L$. Hence

$$[D(a_1), \check{\alpha}^k(a_2)] + (-1)^{Da_1}[\check{\alpha}^k(a_1), D(a_2)] = D'([a_1, a_2]).$$

Thus $D \in \text{QDer}(L)$. Therefore,

$$g - f = \varphi(D) \in \varphi(\text{QDer}(L)) \Rightarrow \text{Der}(\check{L}) \subseteq \varphi(\text{QDer}(L)) + \text{ZDer}(\check{L}).$$

By Proposition 3.2 (3) we have

$$\text{Der}(\check{L}) = \varphi(\text{QDer}(L)) + \text{ZDer}(\check{L}).$$

For all $f \in \varphi(\text{QDer}(L)) \cap \text{ZDer}(\check{L})$, there exists an element $D \in \text{QDer}(L)$ such that $f = \varphi(D)$. Then

$$f(at + bt^2 + ut^2) = \varphi(D)(at + bt^2 + ut^2) = D(a)t + D'(b)t^2,$$

for all $a \in L, b \in [L, L]$.

On the other hand, since $f \in \text{ZDer}(\check{L})$, we have

$$f(at + bt^2 + ut^2) \in \text{Z}(\check{L}) = Lt^2.$$

That is to say, $D(a) = 0$, for all $a \in L$ and so $D = 0$. Hence $f = 0$.

Therefore $\text{Der}(\check{L}) = \varphi(\text{QDer}(L)) \oplus \text{ZDer}(\check{L})$ as desired. \square

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